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A class of integrals associated with the Bethe ansatz

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Abstract. A method is described for evaluating integrals of the form

$$\int_0^\infty \operatorname{sech}\left(\frac{\pi x}{2\mu}\right) \ln\left[\frac{\cosh x - \cos(m\mu)}{\cosh x - \cos(n\mu)}\right] \mathrm{d}x$$

which arise in treating certain statistical mechanical models by means of the Bethe ansatz (quantum inverse scattering) technique. We specify which integrals of this class can be expressed in closed form, along with their values and evaluate some generalizations.

1. Introduction

Integrals of the form

$$I(m,n) = \int_0^\infty \operatorname{sech}\left(\frac{\pi x}{2\mu}\right) \ln\left[\frac{\cosh x - \cos(m\mu)}{\cosh x - \cos(n\mu)}\right] \mathrm{d}x$$

to the authors' knowledge, first appeared in the work of Lieb, Sutherland and others [1–5], examining the free energy for a class of two-dimensional 'vertex' models describing the thermodynamics of hydrogen-bonded ferroelectrics by means of the Bethe ansatz (quantum inverse scattering) technique. The simplest of these ($\mu = 2\pi/3$, $m = 0$, $n = 1$), was evaluated by Lieb [1] to obtain the residual entropy of 'square' ice. Subsequently, the first author evaluated the integrals for all the exactly solvable models of this class [6, 7]. The method was to use Parseval's identity to transform the integrand to x^{-1} times a ratio of hyperbolic functions, with two of them in the denominator. Then hyperbolic identities were invoked, *ad hoc*, to eliminate one of the denominator functions, reducing these integrals to Laplace transforms of a class evaluated in the appendix of [6]. This procedure is specific to precisely the range of μ pertinent to the models, leaving open the value of these integrals outside this parameter range. On re-examining the problem, we have come up with a much simpler technique which provides the values of one-parameter generalizations of these integrals for *all* parameter values. By this means we are able to specify which of the integrals $I(m,n)$ can be evaluated in closed form.

2. Calculation

We shall illustrate our method by applying it to the family of integrals

$$\int_0^\infty \operatorname{sech}\left(\frac{\pi x}{2\mu}\right) \ln\left[\frac{\cosh x - \cos(4k\mu)}{\cosh x - \cos(4l\mu)}\right] \mathrm{d}x \qquad (1)$$

where k and l are integers. Without loss of generality, it is sufficient to restrict attention to the case where $l = 0$. For this purpose, we study the generalization

$$I_k(a) = \int_0^\infty \operatorname{sech}\left(\frac{\pi x}{2\mu}\right) \ln\left[\frac{\cosh x - \cos(4k\mu + a)}{\cosh x - \cos(a)}\right] \mathrm{d}x \tag{2}$$

for $\mu \geqslant 0$, $|a| < \infty$.

First, note the factorization

$$\cosh x - \cos y = \tfrac{1}{2}\mathrm{e}^{-x}\left(\mathrm{e}^x - \mathrm{e}^{\mathrm{i}y}\right)\left(\mathrm{e}^x - \mathrm{e}^{-\mathrm{i}y}\right). \tag{3}$$

Then, by differentiating, we find

$$\frac{\mathrm{d}I_k(a)}{\mathrm{d}a} = \frac{1}{2\mathrm{i}} \int_{-\infty}^\infty \operatorname{sech}\left(\frac{\pi x}{2\mu}\right)[\phi(x) - \phi(x + 4k\mu\mathrm{i})]\,\mathrm{d}x \tag{4}$$

with

$$\phi(x) = \frac{\mathrm{e}^{-\mathrm{i}a}}{\mathrm{e}^x - \mathrm{e}^{-\mathrm{i}a}} + \frac{\mathrm{e}^{\mathrm{i}(4\mu + a)}}{\mathrm{e}^x - \mathrm{e}^{\mathrm{i}(4k\mu + a)}}. \tag{5}$$

We express this as the complex integral

$$\frac{\mathrm{d}I_k(a)}{\mathrm{d}a} = \frac{\pi}{2\pi\mathrm{i}} \oint \operatorname{sech}\left(\frac{\pi z}{2z}\right)\phi(z)\,\mathrm{d}z \tag{6}$$

where the contour is the infinite counterclockwise rectangle bounded by $\mathrm{Im}(z) = 0$, $\mathrm{Im}(z) = 4\mu k$.

The poles of $\operatorname{sech}(\pi z/2\mu)$ inside the contour are $z_n = (2n + 1)\mu\mathrm{i}$ for $0 \leqslant n \leqslant 2k - 1$ with residues $(-1)^n(2\mu\mathrm{i}/\pi)\phi(z_n)$. The two sets of poles for $\phi(z)$, $z_l = -\mathrm{i}(a - 2\pi l)$, $z_m = \mathrm{i}(4k\mu + a - 2\pi m)$ have pairwise identical residues $\sec[\pi(a - 2\pi l)/2\mu]$ $(m = l)$ and lie inside the contour if $a/2\pi < l, m < (4k\mu + a)/2\pi$; we call this condition *. Ignoring the eventuality that in rare cases pairs of poles could coincide, which can always be treated by a suitable limit procedure, the residue theorem gives

$$\begin{aligned}
\frac{\mathrm{d}I_k(a)}{\mathrm{d}a} &= 2\mu\mathrm{i}\sum_{n=0}^{2k-1}(-1)^{n+1}\phi[(2n+1)\mu\mathrm{i}] + 2\pi\sum_l^{*}\sec\frac{\pi}{2\mu}(a - 2\pi l)\\
&= 2\mu\sum_{n=0}^{2k-1}(-1)^{n+1}\cot\left[\tfrac{1}{2}((2n+1)\mu + a)\right] + 2\pi\sum_l^{*}\sec\frac{\pi}{2\mu}(a - 2\pi l).
\end{aligned} \tag{7}$$

Now by integrating over a,

$$I_k(a) = C_k + 4\mu\left[\sum_{n=0}^{2k-1}(-1)^{n+1}\ln\left|\sin\tfrac{1}{2}((2n+1)\mu + a)\right| + \ln\left|\prod_l^{*}\tan\frac{\pi}{2\mu}\left(\tfrac{1}{2}(a+\mu) - \pi l\right)\right|\right]. \tag{8}$$

The integration constant C_k is fixed by the condition $I_k(-2k\mu) = 0$ from which we find that $C_k = 0$. Therefore, we have obtained our principal result

$$\begin{aligned}
&\int_0^\infty \operatorname{sech}\left(\frac{\pi x}{2\mu}\right)\ln\left[\frac{\cosh x - \cos(4k\mu + a)}{\cosh x - \cos(a)}\right]\mathrm{d}x\\
&\quad = 4\mu\left[\sum_{n=0}^{2k-1}(-1)^n\ln\left|\operatorname{cosec}\left[n + \tfrac{1}{2}\mu + \tfrac{1}{2}a\right]\right|\right.\\
&\qquad \left. + \ln\left|\prod_l^{*}\tan\left[\frac{\pi}{2\mu}\left(\tfrac{1}{2}a - l\pi\right) + \tfrac{1}{4}\pi\right]\right|\right].
\end{aligned} \tag{9}$$

3. Discussion

It readily follows from (9) that

$$\frac{1}{4\mu}\int_0^\infty \mathrm{sech}\left(\frac{\pi x}{2\mu}\right)\ln\left[\frac{\cosh x-\cos(4k\mu+a)}{\cosh x-\cos(4p\mu-a)}\right]\mathrm{d}x$$
$$=\sum_{n=0}^{2k-1}(-1)^n\ln\left|\mathrm{cosec}\left[\left(n+\tfrac{1}{2}\right)\mu+\tfrac{1}{2}a\right]\right|-\sum_{n=0}^{2p-1}(-1)^n\ln\left|\mathrm{cosec}\left[\left(n+\tfrac{1}{2}\right)\mu-\tfrac{1}{2}a\right]\right|$$
$$+\ln\left|\frac{\prod_m^{*(k)}\tan\left[(\pi/2\mu)\left(\tfrac{1}{2}a-m\pi\right)+\tfrac{1}{4}\pi\right]}{\prod_n^{*(p)}\tan\left[(\pi/2\mu)\left(\tfrac{1}{2}a+n\pi\right)-\tfrac{1}{4}\pi\right]}\right| \tag{10}$$

for $\mu > 0$, $|a| < \infty$ and integer k, p.

It is clear that the integrals $I(m,n)$ obey the connection formulae

$$\begin{aligned} &I(m,n) = -I(n,m)\\ &I(m,n)+I(n,p) = I(m,p). \end{aligned} \tag{11}$$

The integral $I(2,0)$ is, up to an additive constant, the free energy of the F-model [2], which is known to have an infinite-order phase transition. This integral has been studied in detail in the appendix of [8], where it was shown that it possesses a natural boundary in its analytic structure with respect to μ. This precludes its evaluation in finite terms using standard special functions. The same considerations apply to $I(1,0)$. (We leave the case $I(2,1)$ as an open question.) Therefore, $I(2,0)$ and any integral connected with it by (11) does not have a closed-form expression. (Here we refer to general μ; there may be closed-form expressions for specific values of μ. Just such an evaluation of $I(2,0)$ has been carried out by Adamchik and Ziff [8] in connection percolation on a square lattice, where μ is a multiple of π.) By allowing a to take on as values various multiples of μ in (10) one finds that integrals having closed-form expressions are

$$\begin{aligned} &I(n,n) = 0\\ &I(4k+n,n)\\ &I(4k+2n-1,2n+1)\\ &k = 0,1,2,\ldots \qquad n = 1,2,3,\ldots \end{aligned} \tag{12}$$

and the integrals connected with them. In particular, equation (10) reproduces all the previously known values of these functions.

Since [6]

$$\int_0^\infty \mathrm{sech}(ax)\ln\left[\frac{\cosh(x)-\cos(bx)}{\cosh(x)-\cos(cx)}\right]\mathrm{d}x = \frac{2\pi}{a}\ln\left[\frac{\Gamma\left(ab/2\pi+\frac{3}{4}\right)\Gamma\left(ac/2\pi+\frac{1}{4}\right)}{\Gamma\left(ab/2\pi+\frac{1}{4}\right)\Gamma\left(ac/2\pi+\frac{3}{4}\right)}\right]$$
$$+\frac{2\pi}{a}\int_0^\infty\frac{\mathrm{d}x}{x}\mathrm{e}^{-\pi x}\frac{\sinh(b+c)x/2\,\sinh(c-b)x/2}{\sinh(\pi x)\cosh(\pi x/2a)} \tag{13}$$

we have evaluated a corresponding class of Laplace transforms.

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